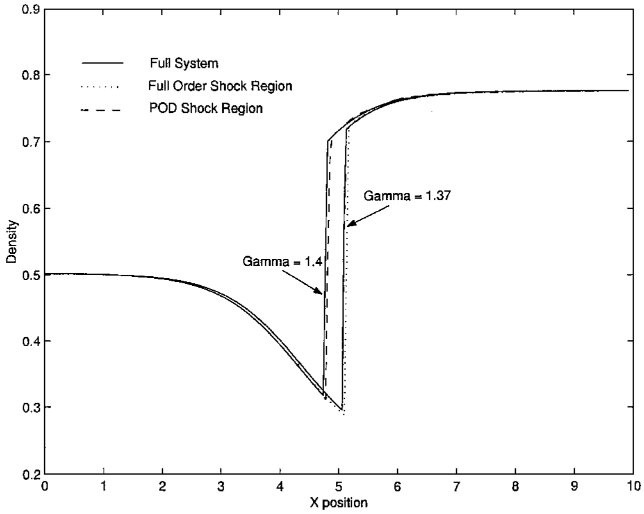

Fig. 3 Shock motion with varying γ.

density results shown in Fig. 3 show that both POD/ROM/DDs tracked the quasi-steady shock motion very closely (within one grid point). In addition, both POD/ROM/DDs accurately solved the flowfield before and after the shock. Density solutions after the shock for both POD/ROM/DDs exhibited a small error growth as gamma was steadily decreased, which occurred because the POD/ROMs for sections I and III were trained at $\gamma = 1.4$. Even with the small error growth, however, the error in density was less than 1%. Errors in density before the shock for both POD/ROM/DDs exhibited a slightly larger error growth (about 2%).

Conclusions

In high-speed flows shock movement can result in the failure of conventional POD/ROM to arrive at a solution. In these cases the POD/ROM will go unstable attempting to form the shock if the shock location is not captured in the snapshots. A new domain decomposition shock-capturing approach was developed to treat moving shocks. The accuracy and order reduction of the domain decomposition approach was demonstrated for quasi-one-dimensional nozzle flow. The nonshocked regions of this flowfield were modeled with POD/ROM trained for $\gamma = 1.4$. The shocked region of the flowfield was modeled both by POD/ROM and by the full-order computational fluid dynamics model adapted for this region. The accuracy of both models was examined for quasi-steady shock motion as γ was varied from 1.4 to 1.37. Both cases produced accurate flowfields and shock motion. Flowfield errors were less than 2%, and the shock movement was tracked within one grid point of the true shock location.

Both methods exhibited similar order reduction. The full-order solution had 750 DOFs, the POD/ROM/DD with a full-order shock region had 58 DOFs, and the POD/ROM/DD with a POD/ROM for the shock region had 55 DOFs. Sixteen modes per fluid variable were required for POD/ROM in the shocked region, resulting in a small-order reduction relative to the full-order shock case. Because of the computational expense of generating snapshots and the large number of modes required, there are no advantages in using POD/ROM for the shocked region for this one-dimensional case. In two- and three-dimensional cases, however, there will be a significant order reduction gained with a POD/ROM for the regions containing shocks.[2,3] The encouraging news from this research is that a small set of DOFs exists that can accurately handle the moving shock case. Future research should focus on efficient methods of solving for these modal coefficients.

References

[1]Holmes, P., Lumley, J., and Berkooz, G., *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*, Cambridge Univ. Press, Cambridge, England, U.K., 1996, pp. 93–107.

[2]Lucia, D. J., "Reduced Order Modeling for High Speed Flows with Moving Shocks," Ph.D. Dissertation, Air Force Inst. of Technology, School of Engineering and Management, Wright–Patterson AFB, OH, Nov. 2001.

[3]Lucia, D. J., Beran, P. S., and King, P. I., "Reduced Order Modeling of an Elastic Panel in Transonic Flow," AIAA Paper 2002-1594, April 2002; also *Journal of Aircraft* (to be published).

[4]Shubin, G., Stephens, A., and Glaz, H., "Steady Shock Tracking and Newton's Method Applied to One-Dimensional Duct Flow," *Journal of Computational Physics*, Vol. 39, 1981, pp. 364–374.

[5]Beran, P., and Silva, W., "Reduced Order Modeling: New Approaches for Computational Physics," AIAA Paper 2001-0853, Jan. 2001.

[6]Lions, J., and Pironneau, O., "Non Overlapping Domain Decomposition for Evolution Operators," *Comptes Rendus de l'Academie des Sciences. Serie I. Mathematique*, Vol. 330, No. 10, 2000, pp. 943–951.

[7]Lions, J., and Pironneau, O., "Overlapping Domain Decomposition for Evolution Operators," *Comptes Rendus de l'Academie des Sciences. Serie I. Mathematique*, Vol. 330, No. 10, 2000, pp. 937–943.

[8]Gunzburger, M. D., and Lee, H. K., "An Optimization-Based Domain Decomposition Method for the Navier–Stokes Equations," *SIAM Journal of Numerical Analysis*, Vol. 37, No. 5, 2000, pp. 1455–1480.

[9]Du, Q., and Gunzburger, M. D., "A Gradent Approach to Optimization-Based Multidisciplinary Simulations and Nonoverlapping Domain Decomposition Algorithms," *SIAM Journal of Numerical Analysis*, Vol. 37, No. 5, 2000, pp. 1513–1541.

[10]Farhat, C., Macedo, A., Lesoinne, M., Roux, F., Magoules, F., and de La Bourdonnaie, A., "Two-Level Domain Decomposition Methods with Lagrange Multipliers for the Fast Iterative Solution of Acoustic Scattering Problems," *Computer Methods in Applied Mechanics and Engineering*, Vol. 184, No. 2, 2000, pp. 213–241.

[11]Strang, G., *Introduction to Applied Mathematics*, Wellesley Cambridge Press, Cambridge, MA, 1986, pp. 96–100.

[12]Lucia, D. J., King, P. I., Beran, P. S., and Oxley, M. E.,"Reduced Order Modeling for a One-Dimensional Nozzle Flow with Moving Shocks," AIAA Paper 2001-2602, June 2001.

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Near-Exact Analytical Solutions of Linear Time-Variant Systems

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Introduction

LINEAR time-variant equations, such as the Mathieu–Hill equation, occur in many applications, such as dynamic buckling of structures and wave propagation in periodic media. Periodic variation of parameters in mechanical devices is also common, such as in the meshing of spur gears.[1] One can come across exponential or hyperbolic functions in the coefficients of the differential equation of motion of cables with varying length. Analytical procedures adopted to solve these types of equations involve complex mathematics, even for a one-dimensional problem.[2] Another instance of an exponential variation in system parameters is in structures with an adaptive nature. For example, the active control of the stiffness of vehicle

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suspensions involves linear time-variant dynamics,[3] wherein the stiffness takes a form $k_{\text{new}} = k_{\text{old}} + [1 - \exp(-t/\Delta t)]\Delta k$. Analytical techniques for single-degree-of-freedom (SDOF) time-variant systems under free vibration can be obtained by using Bessel functions (see Ref. 4). Investigations have been carried out to understand the response of forced systems[5,6] when there are slowly time-varying parameters. Here, the approximation is made that, at a given time instant, the system is frozen in time, and the error associated in each such time step is minimized by formulating an approximate error estimate. Another approach would be to use a stable time integration procedure such as the Wilson–θ method. However, for a rapidly time-varying system, numerical time integration may result in erroneous results because the natural frequency ω_n is time varying, causing the time step chosen in the integration to be too small or too large. The former would lead to increased computational costs and the latter to an increased error.

In general, if the mass of the system is not rapidly varying with time, SDOF systems are represented by the following equation:

$$\ddot{x} + [c(t)/m]\dot{x} + \omega_n^2(t)x = [f(t)/m] \tag{1}$$

where the terms c, ω_n, m, and f refer to the instantaneous damping, natural frequency, mass, and external force, respectively. Here, we use a scheme wherein the state transition matrix that is obtained as an exponential of a matrix can be applied to any linear time-variant system. Both free and forced vibration of time-variant systems are considered here.

Formulation as an Exact Differential Equation

Consider the governing differential equation of a homogenous linear time-variant system written in the state-space form

$$\{\dot{x}\} + [C(t)]\{x\} = 0 \tag{2}$$

This can be expressed as an exact differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left\{\int_0^t [C(t)]\,\mathrm{d}t\right\}\{x\}\right) = 0 \tag{3}$$

if the exponential of the matrix

$$\exp\left\{\int_0^t [C(t)]\,\mathrm{d}t\right\}$$

and $[C(t)]$ commute, that is,

$$\exp\left\{\int_0^t [C(t)]\,\mathrm{d}t\right\}[C(t)] = [C(t)]\exp\left\{\int_0^t [C(t)]\,\mathrm{d}t\right\}$$

Here, $C(t)$ is a matrix of time-varying coefficients. Usually, the only instances when the matrices $C(t)$ and

$$\exp\left\{\int_0^t [C(t)]\,\mathrm{d}t\right\}$$

commute are when $[C]$ is diagonal or when $[C]$ is a matrix of constants or consists of certain trigonometric forms. None of these is the case here. However, as we shall show, Eq. (3) can still be used to describe linear differential equations with time-varying parameters, but within a certain time duration, the duration being dependent on the rate of variation of parameters. We investigate the exactness of the solution to the governing Eq. (2), expressed by Eq. (3), in terms of the rate of variation of the system parameters. Using the solution for this time duration, we can march in time to obtain the response during the entire duration of interest. Thus, we shall arrive at a technique that is applicable to any linearly time-varying system.

Equation (3) requires the evaluation of an exponential of a matrix. When distinct eigenvalues are assumed and Sylvester's theorem is used, a closed-form expression for the exponential of a matrix can be found (see Ref. 7) as

$$\exp[C(t)] = \beta\begin{pmatrix} \dfrac{c_{11} - c_{22}}{2} + \dfrac{\Delta}{\tanh(\Delta)} & c_{12} \\ c_{21} & \dfrac{c_{22} - c_{11}}{2} + \dfrac{\Delta}{\tanh(\Delta)} \end{pmatrix} \tag{4}$$

where $2\Delta = \sqrt{[(c_{11} - c_{22})^2 + 4c_{12}c_{21}]}$ and $\beta = \exp\{[(c_{11} + c_{22})/2][\sinh(\Delta)/\Delta]\}$

Application to the Mathieu Equation

The classical problem of the buckling of a slender beam under periodic loads is governed by the Mathieu–Hill equation

$$\ddot{x} + 2\,\xi\dot{x} + \Omega^2[a - q\cos(\lambda t)]x = 0 \tag{5}$$

where $\Omega^2 = (1/m)(n\pi/l)^2$, $a = EI(n\pi/l)^2$, and q is the amplitude of the periodic external force with a frequency λ applied to the system. Here, ξ is a measure of the system damping, m is the mass of the beam, and n is the buckling mode.

Conventional methods of solving this equation make use of the periodicity of the loading, which allows for the application of the Floquet theory. To illustrate the simplicity and accuracy of the present method, Eq. (5) is written in the state-space form and the matrix $[C]$ in Eq. (3) is given as

$$C = \begin{pmatrix} 0 & -1 \\ \Omega^2[a - q\cos(\lambda t)] & \xi \end{pmatrix}$$

Let us assume $\xi = 0.05$, $a = 0.9975$, $q = -1.9$, $\Omega = 1$, and $\lambda = 0.25$.

After performing the integration,

$$\int_{t_0}^t [C(t)]\,\mathrm{d}t = \begin{pmatrix} 0 & t_0 - t \\ a(t - t_0) - \{(q/\lambda)[\sin(\lambda t)] - \sin[(\lambda t_0)]\} & \xi(t - t_0) \end{pmatrix}$$

the exponential of a matrix is evaluated by means of Eq. (4), and, if initial conditions are $x(0) = 1$ and $\dot{x}(0) = 0$, the solution can be obtained from Eq. (3). Note that the parameters chosen in the example of the Mathieu equation are very close to the unstable region in the Strutt diagram (see Ref. 8). Because we have some damping present, the growth in amplitude is more controlled than for an undamped system. To minimize the error with an increase in time, we divide the time duration of interest into intervals of 0.1 s and use the response at the end of each 0.1-s interval as the initial conditions for the next 0.1-s interval. In other words, the response of the system is governed by the following procedure:

$$\begin{pmatrix} x_{i+1} \\ \dot{x}_{i+1} \end{pmatrix} = \Phi(t)\begin{pmatrix} x_i \\ \dot{x}_i \end{pmatrix} \tag{6}$$

where $\Phi(t)$ is the state transition matrix, calculated for each time interval as

$$\Phi(t_{i+1}, t_i) = \exp\left\{-\int_{t_i}^{t_{i+1}} [C(t)]\,\mathrm{d}t\right\} \tag{7}$$

The result of this procedure is shown in Fig. 1 and is compared with the numerical solution generated by Mathematica, which uses an implicit Adams method. Figure 1 shows that the solution to the Mathieu–Hill equation, as described by Eqs. (3) and (6), is valid. It can be seen from Fig. 1 that the results obtained by the present approach are identical to the reference results. Thus, for the time period considered, $\Delta t = 0.1$ s, the exponential of the matrix as given in Eq. (7) is the state transition matrix, an analytical solution of which needs to be calculated only once and then is applied to small intervals of time. Reference 9 recommends averaging the time-varying parameters over each subdivided time interval. Then, the differential equation, Eq. (5), is solved for each time interval with the time-varying parameter replaced by its average for that time interval. By

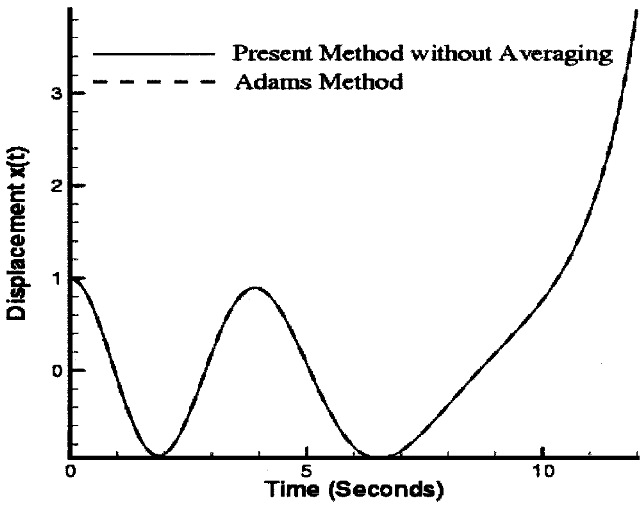

Fig. 1 Comparison of the response of the Mathieu equation $\ddot{x} + 0.1\dot{x} + [0.9975 + 1.9\cos(0.25t)]x = 0$ using matrix exponent time marching without averaging the time-varying parameters with the implicit Adams method.

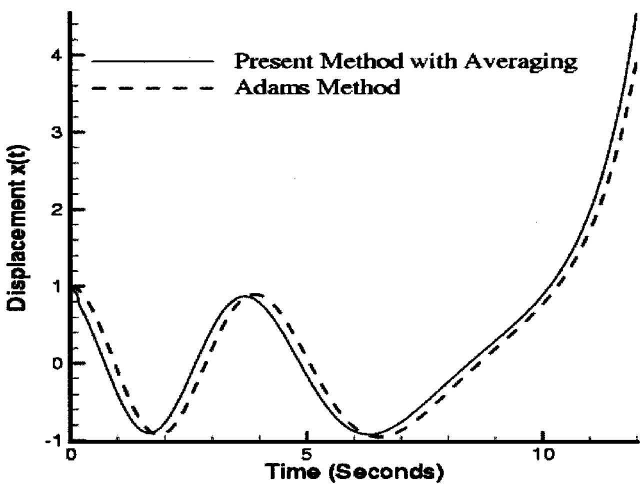

Fig. 2 Comparison of the response of the Mathieu equation $\ddot{x} + 0.1\dot{x} + [0.9975 + 1.9\cos(0.25t)]x = 0$ using matrix exponent time marching with averaging the time-varying parameters over each time interval with the implicit Adams method.

the use of this approach,[9] Eq. (5) was solved using the same times step as before, that is, 0.1 s. One can see from Fig. 2 that the results are not accurate. Therefore, the time-varying parameters should not be averaged over the time interval. Instead, use Eq. (6) with the state transition matrix for each time interval being given by Eq. (7).

Rapidly Time-Variant System

The method of obtaining the response of a linear time-variant system by the use of Eq. (3) is now applied to a system with an exponentially decreasing stiffness. Its nature of variation is similar to the one used by Li.[4] The system is governed by the equation

$$\ddot{x} + \omega_n^2(t)x = 0 \tag{8}$$

where the time-varying natural frequency is given by $\omega_n^2(t) = 40e^{-t}$. The exponential of the matrix in Eq. (3) is then represented as

$$-\int_{t_0}^{t} C(t)\,\mathrm{d}t = \begin{pmatrix} 0 & t - t_0 \\ 40e^{-t} - 40e^{-t_0} & 0 \end{pmatrix}$$

where t_0 is the initial time. The exact solution for this type of a differential equation is given by a summation of Bessel functions (see Ref. 10). By the use of Eqs. (6) and (7) with 0.1-s duration for each time interval, the system governed by Eq. (8) is solved. Figure 3 shows that the present method matches the exact solution very well. Figure 4 compares the result obtained using the method provided in

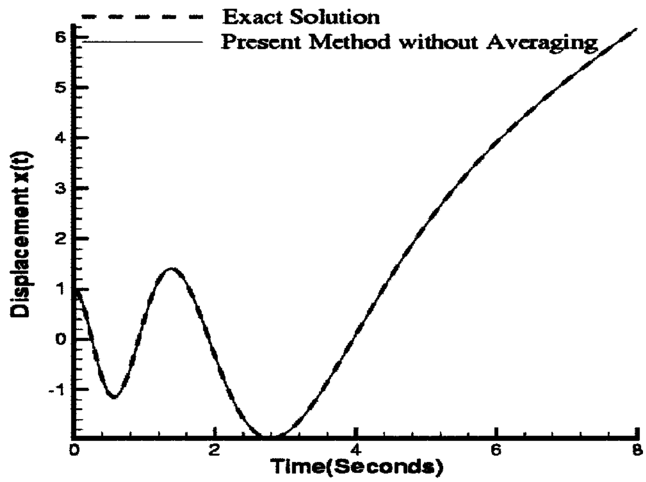

Fig. 3 Comparison of the response of the rapidly varying system $\ddot{x} + 40\exp[-t]x = 0$ using matrix exponent time marching without averaging the time-varying parameters with the exact solution.

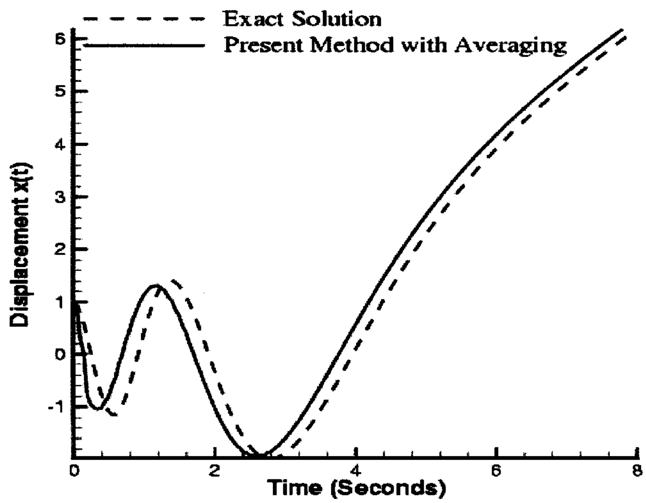

Fig. 4 Comparison of the response of the rapidly varying system $\ddot{x} + 40\exp[-t]x = 0$ using matrix exponent time marching with averaging the time-varying parameters over each time interval with the exact solution.

Ref. 9, wherein the time-varying parameters are averaged over each time interval with the exact solution. It can be seen that the solution obtained using the approach of Ref. 9 gives inaccurate results. We now compare our scheme with the widely used, Newmark–β and Wilson–θ methods. Figure 5 shows the response of the system given by Eq. (8) as calculated by the Newmark–β and Wilson–θ methods with a time step of 0.08 s. These are compared to the exact solution. The present method provides a continuous solution in each time interval of 0.08 s, chosen in the present example, and matches the exact solution. To approach the accuracy of the present method, the time steps needed for the Newmark–β and Wilson–θ methods has to be of the order of 0.002 s, whereas, as shown in Fig. 5, by the use of the time-marching scheme with the exponential of a matrix, an accurate result is obtained even with a time step of 0.08 s, 40 times greater than the time step required by the two widely used methods.

Forced Oscillations of a Time-Variant System

The exact differential equation formulation (3) will be modified as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left\{\int_{t_0}^{t}[C(t)]\,\mathrm{d}t\right\}\{x\}\right) = \exp\left\{\int_{t_0}^{t}[C(t)]\right\}\boldsymbol{Q} \tag{9}$$

where $\boldsymbol{Q}$ is the vector containing the forcing function and t_0 is the time instant at which the initial conditions are defined. The exact solution of the problem will be

$$x(t) = x_h(t) + x_p(t) \tag{10}$$

Table 1 Results for forced vibration using an 0.08-s time step[a,b]

Method	$x(t)$ forced vibration	Error from exact, %
Newmark–β	−73.935	−9.65
Wilson–θ	−70.236	−14.16
Present method	−81.886	0.072

[a]Displacement of mass after 40 s.
[b]Reference solution = −81.8277.

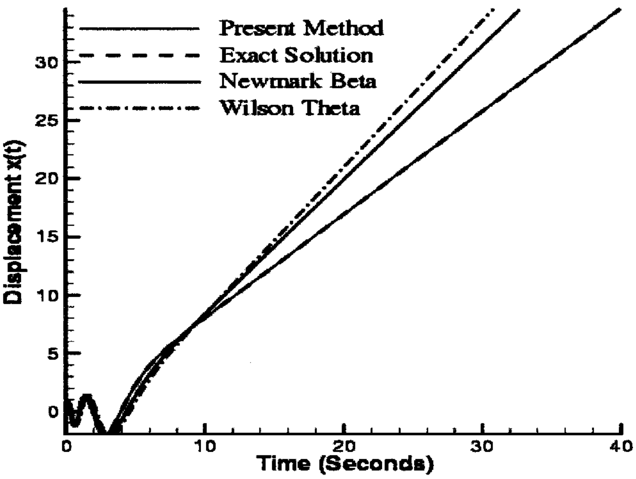

Fig. 5 Comparison of the response of the system $\ddot{x} + 40\exp[-t]x = 0$ using the present method with explicit numerical schemes and the exact solution; time step chosen is 0.08 s.

where $x_h(t)$ and $x_p(t)$ are, respectively, the homogeneous solution and the particular integral. The homogeneous solution is the solution to the free-vibration problem, which has been examined previously in great detail. The particular integral is given by

$$x_p(t) = \exp\left\{\int_{t_0}^{t}[-C(t)]\,\mathrm{d}t\right\}\int_{t_0}^{t}\left(\exp\left\{\int_{t_0}^{t}[C(\tau)]\,\mathrm{d}\tau\right\}Q\right)\mathrm{d}t \tag{11}$$

We consider the same example of Eq. (8), but with a forcing function,

$$\ddot{x} + \omega_n^2(t)x = 2\cos(0.5t) \tag{12}$$

where the time-varying natural frequency is given by $\omega_n^2(t) = 40e^{-t}$. To simplify the particular solution given by Eq. (11), we make the approximations that

$$\lim_{z\to 0}\cosh(z) = 1, \qquad \lim_{z\to 0}\sinh(z) = z$$

With this done, the particular solution can be evaluated analytically in a closed form. This approximation does not cause any discrepancy for the time interval sizes used herein. The procedure for obtaining the response of a time-variant system under forced vibrations is written as

$$\begin{pmatrix} x_{i+1} \\ \dot{x}_{i+1} \end{pmatrix} = \Phi(t).\begin{pmatrix} x_i \\ \dot{x}_i \end{pmatrix} + \begin{pmatrix} x_{p_i} \\ \dot{x}_{p_i} \end{pmatrix} \tag{13}$$

Table 1 shows the comparison between the results obtained using Newmark–β and Wilson–θ methods with the present method. The present method is shown to be much more accurate. This is because an analytical solution was first determined. Hence, this solution will be extremely accurate, provided that the time step selected for time marching is small enough such that the noncommutativity of the matrices involved in the exact differential formulation, Eq. (9), will not cause a discrepancy. For the present example, we have chosen a time step of 0.08 s. The Wilson–θ and the Newmark–β methods were also run using a time step of 0.08 s. The value of the parameter θ in the Wilson method was chosen as 1.4 because this ensures unconditional stability. A measure of the error in the response computed by the present solution can be obtained by noting the result of the noncommutativity of the matrices, that is,

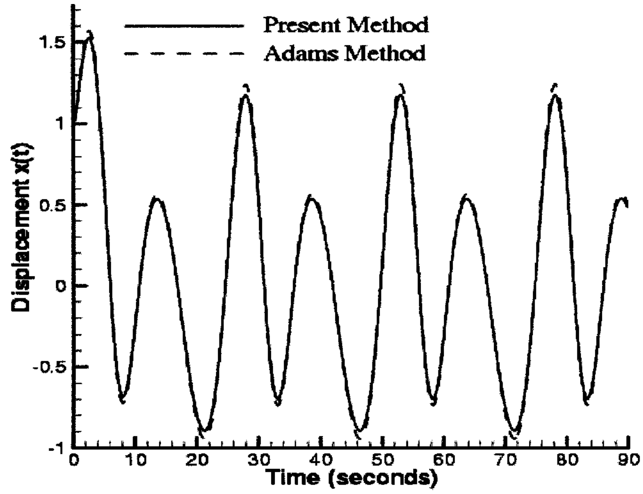

Fig. 6 Response of the forced Mathieu equation $\ddot{x} + 1.8\dot{x} + [1 - 0.8\cos(0.25t)]x = \cos(0.5t)$.

$$\exp\left\{\int_{t_0}^{t}[C(t)]\,\mathrm{d}t\right\}[C(t)] - [C(t)]\exp\left\{\int_{t_0}^{t}[C(t)]\,\mathrm{d}t\right\} = \epsilon(t) \tag{14}$$

Therefore, even if the exact solution to the problem is unknown, an estimate of the error is obtained as a function of time. This can be used to estimate the time step required. Shahruz and Tan[6] give an example of the Mathieu–Hill equation:

$$\ddot{x} + 1.8\dot{x} + [1 - 0.8\cos(0.25t)]x = \cos(0.5t)$$

The authors of Ref. 6 used a "time freezing" technique at particular instants to evaluate the solution, but this approximation did not provide satisfactory results. Using the technique described here, of time marching with an approximation to the state transition matrix in the form of the exponential of a matrix [Eq. (7)], we obtain the result shown in Fig. 6. Though there is a slight error between the present solution and the exact solution as the response reaches the peak amplitude, the present method does provide a very accurate fit. The present solution is also consistent for large time because the error with respect to the reference solution does not grow with time, as can be seen from Fig. 6.

Conclusions

Linear time-variant systems have been analyzed analytically using Eqs. (6), (7), and (11), and their response has been obtained in a closed form, within a time interval, in the form of the exponential of a matrix. The exponential of a matrix can be evaluated using the closed-form expression given in Eq. (4). The time duration of interest is divided into intervals in which the closed-form solution is valid, and, by successive application of the exponential of the matrix for many such intervals, the closed-form solution can be applied for any time length. The time interval used in this time marching is much higher compared to traditional numerical schemes such as the Runge–Kutta, Newmark–β, or Wilson–θ methods. The present methodology proved far more accurate than the Newmark–β or Wilson–θ methods and was shown to be consistent with implicit numerical methods. This technique can be applied to free and forced oscillations, and various examples were successfully analyzed, such as the damped Mathieu oscillator and a system with exponentially varying natural frequency. The present methodology provides a continuous solution in time because it is based on an analytical solution, unlike numerical solutions that are evaluated at discrete time steps.

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References

[1]Amabili, M., and Rivola, A., "Dynamic Analysis of Spur Gear Pairs: Steady State Response and Stability of the SDOF System with Time Varying Mesh Damping," *Mechanical Systems and Signal Processing*, Vol. 11, No. 3, 1997, pp. 375–390.

[2]Kotera, T., and Kawai, R., "Vibrations of Strings with Time-Varying Length—(The Case Having a Weight at One End)," *Transactions of the Japan Society of Mechanical Engineers*, Vol. 31, No. 3, 1988, pp. 524–529.

[3]Youn, I., and Hao, A., "Semi Active Suspensions with Adaptive Capability," *Journal of Sound and Vibration*, Vol. 180, No. 3, 1995, pp. 475–492.

[4]Li, Q. S., "A New Exact Approach for Analyzing Free Vibration of SDOF Systems with Nonperiodically Time Varying Parameters," *Journal of Vibration and Acoustics*, Vol. 122, No. 2, 2000, pp. 175–179.

[5]Liu, K., and Kujath, M. R., "Response of Slowly Time Varying Systems to Harmonic Excitation," *Journal of Sound and Vibration*, Vol. 177, No. 3, 1994, pp. 423–432.

[6]Shahruz, S. M., and Tan, C. M., "Response of Linear Slowly Varying Systems Under External Excitations," *Journal of Sound and Vibration*, Vol. 131, No. 2, 1989, pp. 239–247.

[7]Douglas, J. M., *Process Dynamics and Control, Volume 1, Analysis of Dynamic Systems*, Prentice–Hall, Upper Saddle River, NJ, 1972.

[8]Nayfeh, A. H., and Mook, D. T., *Nonlinear Oscillations*, Wiley-Interscience, New York, 1979, pp. 277, 278.

[9]Hsu, C. S., "On Approximating a General Linear Periodic System," *Journal of Mathematical Analysis and Applications*, Vol. 45, No. 2, 1974, pp. 234–251.

[10]Cheng, H., and Wu, T. T., "An Aging Spring," *Studies in Applied Mathematics*, Vol. 49, No. 2, 1970, pp. 183–185.

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Predicting Ballistic Penetration and the Ballistic Limit in Composite Material Structures

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Introduction

THE ability to predict whether a given ballistic blunt object will penetrate a composite material structure is valuable. If penetration occurs the ability to predict the residual velocity of the object is also valuable. Also of interest is the ability to predict the ballistic limit of any projectile–composite structure combination.

Much work has been done in the field of analyzing the ballistic impact of composite targets; however, most of that work has been empirical in nature. Vinson and Zukas[1] introduced the use of conical shell theory as an analytical tool for determining the ballistic properties of woven Kevlar® targets. Then Vinson and Walker tested this model on fiber-reinforced composite targets.[2] The link between woven fabrics and composites was the assumption that if a matrix material contributed an insignificant amount to a fiber-reinforced composite's strength and strain properties then the entire structure would behave similarly to a woven fabric.

Vinson and Walker[2] utilized Lee and Sun's data[3] to test this analytical method and also compared the results with Ref. 3. At that time, the results seemed conclusive, but subsequent review of Ref. 2 has revealed that, although the underlying theory was correct, a computational error led to incorrect numerical results. This Note offers an improved method for determining the ballistic limit and revalidates the conical shell method for predicting if penetration will occur or not and if so what the residual velocity will be. In addition to validating the method against the data utilized in Ref. 2, more experiments were analyzed using data from other authors including Sun and Potti [4] and Silva et al.[5] The findings from these sources contribute to the validity of the model and also lead to additional conclusions about the conical shell method.

Analysis Methods

When a ballistic projectile impacts a composite plate, as shown in Fig. 1 in Ref. 2, a conical shell forms and continues to propagate until either the projectile velocity is reduced to zero or until the projectile penetrates the target material. Vinson and Walker[2] have shown that the conical shell is primarily in a state of membrane stress and that the resistance to penetration is mostly due to membrane strain energy. At the front of the conical shell is a radius R_1, which is determined by the dimensions of the blunt projectile.

The base radius of the shell is determined from the initial radius along with the propagation of a shear wave over time and is given as $R_2 = R_1(t = 0) + C_s t$. In this equation, C_s is the shear wave velocity taken to be $\sqrt{(G_{yz}/\rho_m)}$, where y is the meridional coordinate of the shell and z is the coordinate in the shell thickness direction. G_{yz} is the dynamic modulus at the given strain rate occurring; however, if that is not available, then the static value should be used.

Vinson and Walker[2] describe an iterative method to calculate the conical shell parameters and relate them to the ability of a composite target to capture a blunt projectile. When the required material properties and a striking velocity are supplied to an iterative solver, a time history is created that details the strain and projectile velocity at each time step. Test data consisting of striking velocity and residual velocity data points can then be used to correlate striking velocity to the ultimate strain at failure. This is accomplished by entering a given striking velocity and finding the time step that has a residual velocity that matches the experimental data point. The strain in the conical shell at that time step is determined to be the ultimate strain for that data point. The strain-striking velocity points are then plotted, and a linear regression is performed to determine the equation of the line of best fit for the data. This equation, then, relates striking velocity to maximum shell strain at failure. Examples of this are shown in Fig. 1.

The method used to determine the ballistic limit differs from that found in Ref. 2 and is considered the valid method for applying

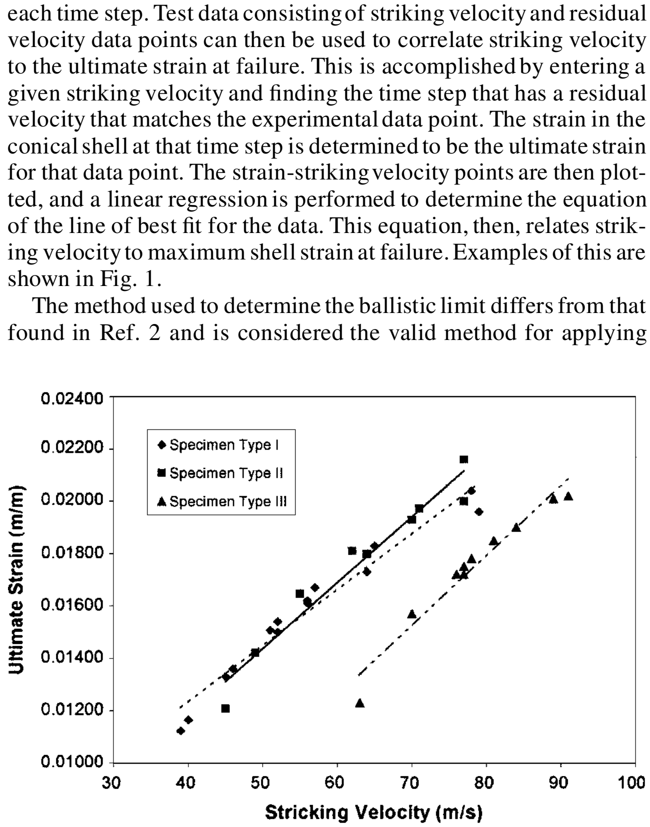

Fig. 1 Maximum strain to failure as a function of striking velocity for data of Ref. 3.

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